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CENTRAL EXTENSIONS OF THE SMALL-DIMENSIONAL SOLVABLE LIE ALGEBRA

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Abstract
In this thesis, it is necessary to consider the filtered cohomology structure relative to the ideals of the lower central series: a cocycle defining the central extension is needed has maximum filtration. Such a geometric method allows us to classify nilpotent Lie algebras small dimensions, as well as for classifying Lie algebras of narrow natural rank. Concept a rigid central extension is introduced. Examples of rigid and non-rigid central extensions built.

Keywords: Lie algebra, Rigid central extensions of nilpotent Lie algebras, central extensions of lie algebras

INTRODUCTION

An arbitrary nilpotent Lie algebra is a central extension of a nilpotent Lie algebra of lower dimension. Question: Is it possible to organize a recurrent procedure using such a construction and classify finite-dimensional nilpotent Lie algebras?

The very first analysis of the posed question shows its transcendental complexity, the answer to it is hardly accesible in a general setting and for an arbitrary dimension, but in small dimensions or for some special classes of nilpotent Lie algebras, answers can still be obtained.

We start the study with small dimensions. According to Morozov's well-known classification, in dimensions ≤ 6 there exists a finite number of pairwise non-isomorphic nilpotet Lie algebras over a field of characteristic zero. Staπrting with the dimension 7 (where a one-parameter family of pairwise non-isomorphic nilpotent Lie algebras appears), the difficulties of classifying nilpotent Lie algebras are rapidly increasing, which leads, in particular, to the need to consider the so-called affine $\mathbf{L_n}$ variety of Lie algebra structures on a fixed $\mathbf{n}$-dimensional vector space $\mathbf{V}$ over the field $\mathbf{K}$. The manifold $\mathbf{L_n}$ consists of skew-symmetric bilinear mappings $\boldsymbol{\mu} : \mathbf{V} \wedge \mathbf{V} \to \mathbf{V}$ satisfying the Jacobi identity. The affine variety $\mathbf{N_n}$ of nilpotent Lie algebras is also defined. There is a natural $\mathbf{GL_n}$-action on $\mathbf{L_n}$ (respectively on $\mathbf{N_n}$:

(1.1) $$(\mathbf{g} \cdot \boldsymbol{\mu})(\mathbf{x}, \mathbf{y}) = \mathbf{g}\,(\boldsymbol{\mu}(\mathbf{g}^{-1}\mathbf{x}, \mathbf{g}^{-1}\mathbf{y})) \ , \mathbf{g} \in \mathbf{GL_n},\ \mathbf{x}, \mathbf{y} \in \mathbf{V}.$$

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Obviously, the isomorphism class of a given algebra (structure) of Lie $\mu \in L_n$ corresponds to the orbit $O(\mu)$ of this action.

CENTRAL EXTENSIONS OF LIE ALGEBRAS

Central extension of Lie algebra **g** is called the exact sequence

(1.2) $$0 \to V \overset{i}{\to} \tilde{g} \overset{\pi}{\to} g \to 0$$

Lie algebras and their homomorphisms, in which the image of the homomorphism $i : V \to \tilde{g}$ is contained in the center $Z(\tilde{g})$ of the Lie algebra $\tilde{g}$, and the linear subspace **V** is considered as an abelian Lie algebra.

Example 1. The Lie algebras $m_1(2m-1)$ and $m_0(2m-1)$ are one-dimensional central extensions of the Lie algebra $m_0(2m-2)$ for $m \geq 3$. As a vector space, the central extension $\tilde{g}$ is a direct sum $V \oplus g$ with standard inclusion **i** and projection π. The Lie bracket in the vector space $V \oplus g$ can be defined by the formula

(2.1) $$[(v, g),(w, h)]_{\tilde{g}} = (c(g, h), [g, h]_g), \qquad g, h \in g,$$

where c is a skew-symmetric bilinear function on **g**, which takes its values in the **space = V** , and $[\cdot, \cdot]_g$ defines the Lie bracket of a Lie algebra g. One can verify directly that the Jacobi identity for the bracket $[\cdot, \cdot]_{\tilde{g}}$ is equivalent to the condition that the bilinear function is a cocycle, **i**.e. the following equality holds identically

$$c([g, h]g, e) + c([h, e]g, g) + c([e, g]g, h) = 0, \forall g, h, e \in g,$$

we assume that the initial bracket $[g, h]_g$ satisfies the Jacobi identity.

Two extensions are called equivalent if there is an isomorphism of Lie algebras $f : \tilde{g}_2 \to \tilde{g}_1$, such that the following diagram is commutative

(2.2)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i_1} & \tilde{\mathfrak{g}}_1 & \xrightarrow{\pi_1} & \mathfrak{g} & \longrightarrow & 0 \\ \uparrow & & \uparrow Id & & \uparrow f & & \uparrow Id & & \uparrow \\ 0 & \longrightarrow & V & \xrightarrow{i_2} & \tilde{\mathfrak{g}}_2 & \xrightarrow{\pi_2} & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

A cocycle c is called cohomologous to zero $c \sim 0$ if such a linear mapping exists $\mu : g \to V$ such that $c(x, y) = \mu([x, y]_g)$. In this situation, the cocycle c is called a coboundary and is denoted by $c = d\mu$.

Two cocycles are called cohomologous $c \sim c'$ if their difference is cohomologous to zero $c - c' \sim 0$. Cohomologous cocycles define equivalent central extensions. To prove this, it suffices to verify that the linear mapping

(2.3) $$f = Id + \mu : V \oplus g \to V \oplus g, \quad f(v, g) = (v+\mu(g), g),$$

is an isomorphism of Lie algebras in the diagram (2.2). The converse is also true.

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Note also that the cocycle **$c' \sim 0$** cohomologous to zero defines an extension **$\tilde{g}'$** isomorphic to the direct sum of **$V \oplus g$** Lie algebras. Such a central extension is called trivial.

Remark 1. It may well happen that the Lie algebras $\tilde{g}_2$ and $\tilde{g}_1$, corresponding to nonequivalent central extensions, are nevertheless isomorphic.The fact is that an isomorphism f from a commutative diagram (2.2) has to map **$i_2(V) \subset \tilde{g}_2$ to $i_1(V) \subset \tilde{g}_1$** and induce identity mapping of quotient algebras **$Id : \tilde{g}_2/i_2(V) \rightarrow \tilde{g}_1/i_1(V)$**. The absence of an isomorphism of f with such additional properties does not mean the absence of isomorphism in general. In the general case, an isomorphism is not required to translate **$i_2(V)$** into **$i_1(V)$** However, in the case of a nilpotent Lie algebra **g**, the answer to the question of the isomorphism of its two different central extensions **$\tilde{g}_2$** and **$\tilde{g}_1$** is quite possible and constructive with some **c_1** and **c_2**,that we will show it the section .

Rigid central extensions of nilpotent Lie algebras

Definition. Set of cohomology classes **$\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_m)$** from the space **$(H^2(g, K))^m$** of nilpotent Lie algebra g is called geometrically rigid if such a neighborhood exists **$U(\tilde{c}) \subset (H^2(g, K))^m$** (in the standard topology of a finite-dimensional space) that for any other set of cocycles **$\tilde{c}'$** from this neighborhood the corresponding Lie algebra **$g_{\tilde{c}'}$** constructed as a central extension g over the set **$\tilde{c}'$** over the set **$\tilde{c}'$** will be isomorphic to a Lie algebra **$g_{\tilde{c}}$.** We will immediately clarify that in algebraic literature more often, when it comes to the orbits of an algebraic group, the Zarissky topology is considered and usually the openness of the orbit is understood precisely in the sense of this topology. We will now use an equivalent geometric approach and, accordingly, consider the standard Euclidean topology of a finite-dimensional space to visually describe the orbit spaces of the actions we need for the algebraic subgroups of **GL_2** on some cohomology spaces **$H_2(g, R)$ m** of small dimensions – a similar geometric approach was considered in. It is the real classification that is our main goal, in the light of its various geometric applications. The study of the orbit space of the action of an algebraic group on an affine variety is the subject of the classical theory of invariants, but the goal of this article is more modest: we want to depict orbits that are interesting to us using images and means of elementary low-dimensional geometry. Since the natural action of the group **Aut(g)** on the two-dimensional cohomology space **$H_2(g, K)$** is algebraic, the following statement is true. **Proposition.** Let the orbit space of the action **$GL_m \times Aut(g)$** on the space **$(H2(g, K)^m$** be a finite set. Then there is at least one rigid set of cocycles **$\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_m)$.**

We begin the study of examples from the simplest case. Every non-abelian three-dimensional nilpotent Lie algebra is metabelian and can be obtained as a one-dimensional central extension of a two-dimensional abelian algebra $\mathbf{m_0(1) = (e_1, e_2)}$. Its cocycle $\mathbf{e^1 \wedge e^2}$ spans the intire space $\mathbf{H* (m(1))}$.Automorphism group $\mathbf{Aut(m(1)) = GL_2}$ acts on the line $\mathbf{H^2 (m(1)) = (e^1 \wedge e^2}$) as multiplication by the determinantdet $\mathbf{A, A \in GL_2}$ pf the matrix $\mathbf{A}$ of the corresponding automorphism. The orbits of such an action will be only two: 1) single-point, consisting of the zero cohomology class; 2) an open orbit consisting of a complement to zero on the number line. Thus the cocycle $\mathbf{e_1 \wedge e_2}$ for the Lie algebra $\mathbf{m_0(1)}$ is geometrically rigid and corresponding central extension $\mathbf{m_0(2)}$ commonly called the three-dimensional Heisenberg Lie algebra $\mathbf{h_3}$.The latter is isomorphic to the Lie algebra of strictly upper triangular matrices of order three and can be defined using the basis e1, e2, e3 and one non-trivial commutation relation $\mathbf{[e_1, e_2] = e_3}$ (the remaining commutation relations have the form $\mathbf{[e_i , e_j] = 0)}$. As a methodical corollary, we have obtained the well-known classification of three-dimensional nilpotent Lie algebras, up to isomorphism, there are only two: 1) an abelian Lie algebra and 2) a three-dimensional Heisenberg Lie algebra $\mathbf{h_3}$. We can now continue the process of central extensions and consider the extension of the Heisenberg algebra$\mathbf{h_3}$

REFERENCES

1. V.V. Morosov, Classification of nilpotent Lie algebras of order 6, Izv. Vyssh. Uchebn. Zaved. Mat., 4 (1958), 161–171.
A. Fialowski, Deformations of Lie algebras, Math. Sb., 55:2 (1986), 467–473
2. J. Dixmier, Cohomologie des algebres de Lie nilpotentes, Acta Sci. Math. Szeged, 16 (1955), 246–250
3. L.Yu. Galitski, D.A. Timashev, on classification of metabelian Lie algebras, Journal of Lie Theory, 9 (1999) 125-156
4. D.V. Millionshchikov, Filiform N-graded Lie algebras, Russian Math. Surveys, 57:2 (2002), 422–424.