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A DOUBLE INTEGRAL AS GENERALIZATION OF A DEFINITE INTEGRAL

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Abstract
The ways of calculating of "integrating functions of several variables" in the course of higher mathematics were researched and discussed on this article. However, formulas for calculating repeated (or double) integrals are given. As a sample, some formulas and applying tasks were given to improve the skills.

Keywords: Double integral, repeated integral, limit, interpretation.

A double integral can also be thought of as generalization of a definite integral to a function of two variables.
Recall that we think of the integral in two different ways. In one way we interpret it as the area under the graph $y = f(x)$, while the fundamental theorem of the calculus enables us to compute this using the process of "anti-differentiation" - undoing the differentiation process. We think of the area as

$$\sum f(x_i)dx_i = \int f(x)dx,$$

where the first sum is thought of as a limiting case, adding up the areas of a number of rectangles each of height $f(x_i)$, and width dx_i. This leads to the natural generalization to several variables: we think of the function $z = f(x, y)$ as representing the height of f at the point (x, y) in the plane, and interpret the integral as the sum of the volumes of a number of small boxes of height $z = f(x, y)$ and area dx_i dy_j. Thus the volume of the solid of height $z = f(x, y)$ lying above a certain region R in the plane leads to integrals of the form

$$\iint_R \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)dx_i dy_j = \lim S_{mn}.$$

We write such a double integral as

$$\iint_R f(x,y)dA.$$

Repeated Integrals and Fubini's Theorem. As might be expected from the form, in which we can sum over the elementary rectangles dx dy in any order, the order does not matter when calculating the answer. There are two important orders - where we first keep x constant and vary y, and then vary x; and the opposite way round. This gives rise to the concept of the repeated integral, which we write as

$$\int\left(\int_R f(x,y)dx\right)dy \quad \text{or} \quad \int\left(\int_R f(x,y)dy\right)dx.$$

Our result that the order in which we add up the volume of the small boxes doesn't matter is the following, which also formally shows that we evaluate a double integral as any of the possible repeated integrals [1].

Theorem 1 (Fubini's theorem for Rectangles). Let $f(x, y)$ be continuous on the rectangular region R : $\{a \le x \le b; c \le y \le d\}$. Then

$$\iint_R f(x,y)dA = \int_c^d\left(\int_a^b f(x,y)dx\right)dy = \int_a^b\left(\int_c^d f(x,y)dy\right)dx$$

Note that this is something like an inverse of partial differentiation. In doing the first inner (or repeated) integral, we keep y constant, and integrate with respect to x. Then we integrate with respect to y. Of course if f is a particularly simply function, say $f(x, y) = g(x)h(y)$, then it doesn't matter which order we do the integration, since

$$\iint_R f(x,y)dA = \int_a^b g(x)dx \int_c^d h(y)dy.$$

We use the Fubini theorem to actually evaluate integrals, since we have no direct way of calculating a double (as opposed to a repeated) integral.

Example 1. Integrate $z = 4 - x - y$ over the region $0 \le x \le 2$ and $0 \le y \le 1$. Hence calculate the volume under the plane $z = 4 - x - y$ above the given region.

Solution. We calculate the integral as a repeated integral, using Fubini's theorem.

$$V = \int_{x=0}^{2}\int_{y=0}^{1}(4 - x - y)dydx = \int_{x=0}^{2}[(4y - xy - y^2/2)]_0^1 dx = \int_{x=0}^{2}(4 - x - 1/20dx \text{ etc.}$$

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From our interpretation of the integral as a volume, we recognize V as volume under the plane $z=4-x-y$ which lies above $\{(x, y) \mid 0\le x\le 2; 0\le y\le 1\}$.

Theorem 2 (Fubini's Theorem-Stronger Form). Let $f(x, y)$ be continuous on a region R

- if R is defined as $a\le x\le b; g_1(x)\le y\le g_2(x)$.Then

$$\iint_R f(x,y)dA=\int_a^b\left(\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right)dx$$

- if R is defined as $c\le y\le d; h_1(y)\le x\le h_2(y)$. Then

$$\iint_R f(x,y)dA=\int_c^d\left(\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right)dy$$

Proof:We give no proof, but the reduction to the earlier case is in principle simple; we just extend the function to be defined on a rectangle by making it zero on the extra bits. The problem with this as it stands is that the extended function is not continuous. However, the difficulty can be fixed.

This last form enables us to evaluate double integrals over more complicated regions by passing to one of the repeated integrals [2].

Example 2. Evaluate the integral

$$\int_1^2\left(\int_1^2 \frac{y^2}{x^2}dy\right)dx$$

as it stands, and sketch the region of integration.

Reverse the order of integration, and verify that the same answer is obtained.

Solution. The diagram in a figure shows the area of integration.

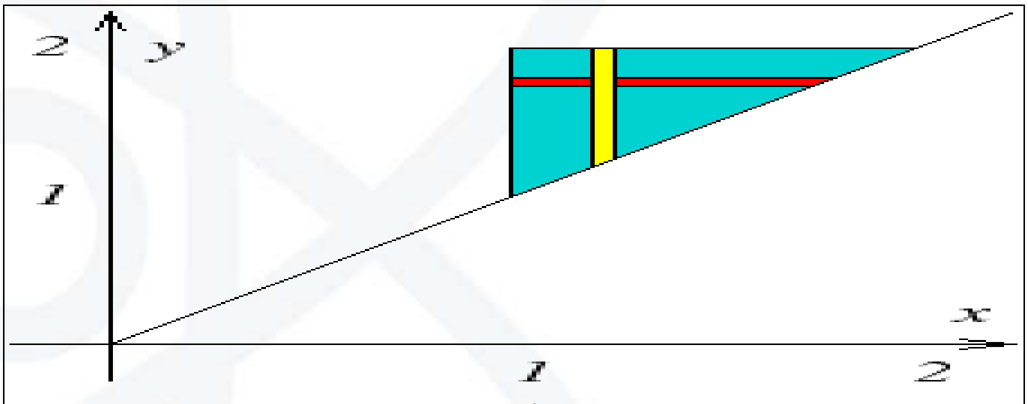

Figure. Area of integration.

We first integrate in the given order.

$$\int_1^2\left(\int_x^2 \frac{y^2}{x^2}dy\right)dx=\int_1^2\left[\frac{y^3}{3x^2}\right]_x^2 dx=\left[-\frac{8}{3x}-\frac{x^2}{6}\right]_1^2=\left(-\frac{4}{3}-\frac{2}{3}\right)-\left(-\frac{8}{3}-\frac{1}{6}\right)=\frac{5}{6}.$$

Reversing the order, using the diagram, gives

$$\int_1^2\left(\int_1^y \frac{y^2}{x^2}dx\right)dy=\int_1^2\left[-\frac{y^2}{x}\right]_1^y dy=\int_1^2\left(-y+y^2\right)dy=\left[-\frac{y^2}{2}-\frac{y^3}{3}\right]_1^2=\left(-2+\frac{8}{3}\right)-\left(-\frac{1}{2}+\frac{1}{3}\right)=\frac{5}{6}.$$

Thus the two orders of integration give the same answer.

Literature

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