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RELATIONS BETWEEN FAITHFUL NONDEGENERATE W* REPRESENTATIONS AND THE CANONICAL REPRESENTATION

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Abstract
In the paper considers the canonical representation of a finite real factor and its commutant. The relation between the canonical representation and its commutants is given.

Keywords: W*-algebra, real factor, canonical representation, commutant, index of real factors.

Introduction

In 1930's von Neumann and Murray introduced the notion of the coupling constant for finite factors (see [11-13]). In 1983, V. Jones suggested a new approach to this notion, defined the notion of the index for type II_1 factors, and proved a surprising theorem on values of the index for subfactors (see [4]). He also introduced a very important technique in the proof of this theorem: the towers of algebras. Since then this theory has become a focus of many fields in mathematics and physics ([5]). In [6], H. Kosaki extended the notion of the index to an arbitrary (normal faithful) expectation from a factor onto a subfactor. While Jones' definition of the index is based on the coupling constant, Kosaki's definition of the index of an expectation relies on the notion of spatial derivatives due to A. Connes [2] as well as on the theory of operator-valued weights due to U. Haagerup [3]. In [6,7] it was shown that many fundamental properties of the Jones' index in the type II_1 case can be extended to the general setting. At present the theory of index thanks to works by V. Jones, P. Loi, R. Longo, H. Kosaki and other mathematicians is deeply developed and has many applications in the theory of operator algebras and physics (see also [9,10]).

Unlike to the complex case, for real factors the notion of the coupling constant (therefore the notion of the index as well) has not been investigated. In the present paper the notions of the real coupling constant and the index for finite real factors are introduced and investigated. The main tool in our approach is the reduction of real

Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. A weakly closed -subalgebra $\mathfrak{A}$ containing the identity operator II $B(H)$ is called a W^*-algebra. A real *-subalgebra $\mathfrak{R} \subset B(H)$ is called a real W^*-algebra if it is closed in the weak operator topology and $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$. A real W-algebra $\mathfrak{R}$ is called a real factor if its center $Z(\mathfrak{R})$ consists of the elements $\{\lambda\mathbf{1}, \lambda \in \mathbb{R}\}$. We say that a real W^*-algebra $\mathfrak{R}$ is of the type $I_{fin}, I_\infty, II_1, II_\infty$, or $III_\lambda, (0 \leq \lambda \leq 1)$ if the enveloping W^*-algebra $\mathfrak{A}(\mathfrak{R})$ has the corresponding type in the ordinary classification of W *-algebras. A linear mapping α of an algebra into itself with $\alpha(x^*) = \alpha(x)^*$ is called an *-automorphism if $\alpha(xy) = \alpha(x)\alpha(y)$; it is called an involutive *-antiautomorphism if $\alpha(xy) = \alpha(y)\alpha(x)$ and $\alpha^2(x) = x$. If α is an involutive *-antiautomorphism of a W^*-algebra M, we denote by (M, α) the real W^*-algebra generated by α, i.e. $(M, \alpha) = \{x \in M: \alpha(x) = x^*\}$. Conversely, every real W^*-algebra $\mathfrak{R}$ is of the form (M, α), where M is the complex envelope of $\mathfrak{R}$ and α is an involutive *-antiautomorphism of M (see [1,5,14]). Therefore we shall identify from now on the real von Neumann algebra $\mathfrak{R}$ with the pair (M, α)

Canonical Representation

Let $M(\subset B(H))$ be a finite factor and let τ be the unique faithful normal tracial state of M. If α is an involutive *-antiautomorphism of M, then it is clear that τ is automatically α-invariant. Denote by $L^2(M)$ the completion of M with respect to the norm $\| x \|_2 = \tau(x^*x)^{1/2}$. Similarly by $L^2(M, \alpha)$ we denote the completion of the real factor (M, α). Then it is obvious that the Hilbert space $L^2(M)$ and the algebra $B(L^2(M))$ of all bounded linear operators on it are the complexifications of the real Hilbert space $L^2(M, \alpha)$ and of $B_r(L^2(M, \alpha))$, respectively, where $B_r(L^2(M, \alpha))$ is the algebra of all bounded linear operators on the real Hilbert space $L^2(M, \alpha)$. Moreover, it is easy to show that the Hilbert spaces $L^2(M, \alpha)$ and $L^2(M)$ are separable.

For each $x \in M$, set $\lambda(x)y = xy$, for all $y \in M$. Clearly, $\| \lambda(x)y \|_2 \leq \| x \| \| y \|_2$. Thus λ can be uniquely extended to a bounded linear operator on $L^2(M)$, still denoted by $\lambda(x)$. Then we obtain a faithful W^*-representation $(\lambda, L^2(M))$ of M. In a similar way, taking the map λ_r defined as $\lambda_r(x)y = xy$ (for all $x, y \in (M, \alpha)$) we obtain a faithful real *-representation $(\lambda_r, L^2(M, \alpha))$ of (M, α).

Theorem 1

The map $\beta: \lambda(M) \to \lambda(M)$ defined as $\beta(\lambda_x) = \lambda_{\alpha(x)}$ is an involutive *-antiautomorphism of $\lambda(M)$. Moreover, β and α are also related in the following way: $(M, \alpha)_\beta = \lambda_r(M, \alpha)$, where $(M, \alpha)_\beta = \{\lambda_x \in \lambda(M): \beta(\lambda_x) = \lambda_x^*\}$ is the real W^*-algebra, generated by β, i.e. $(M, \alpha)_\beta = (\lambda(M), \beta)$.

Proof

The first part of the assertion is trivial. Further, let $\lambda_x \in (M, \alpha)_\beta$. Since $\beta(\lambda_x) = \lambda_x^*$, then $\lambda_{\alpha(x)} = \lambda_{x^*}$. Hence $\alpha(x) = x^*$, i.e. $x \in (M, \alpha)$. Then from

$$\lambda_x \in \lambda(M) \subset B(L^2(M)) = B_r(L^2(M, \alpha)) + iB_r(L^2(M, \alpha))$$

we have $(M, \alpha)_\beta \subset B_r(L^2(M, \alpha))$. Hence $(M, \alpha)_\beta \subset \lambda_r(M, \alpha)$, since $\lambda_r(M, \alpha) = \{\lambda_x^r \in B_r(L^2(M, \alpha))$: for $\alpha(x) = x^*$ and $\alpha(y) = y^*, \lambda_x^r(y) := xy\}$.

Now let $\lambda_x^r \in \lambda_r(M, \alpha)$. Then $\alpha(x) = x^*$ and $\lambda_x^r \in \lambda_r(M, \alpha) \subset \lambda(M)$. Hence $\beta(\lambda_x^r) = \lambda_{\alpha(x)}^r = \lambda_{x^*}^r = (\lambda_x^r)^*$, therefore $\lambda_x^r \in (M, \alpha)_\beta$.

Corollary 1

$\lambda_r(M, \alpha)$ is a real W-algebra, and $\lambda(M)$ is the complexification of $\lambda_r(M, \alpha)$, i.e. $\lambda_r(M, \alpha) + i\lambda_r(M, \alpha) = \lambda(M)$. Moreover, $\{\lambda_r, L^2(M, \alpha)\}$ is a faithful real W^*-representation of (M, α).

This representation will be called the canonical W^*-representation of (M, α).

Commutant of The Canonical Representation

Since $\| x \|_2 = \|x^*\|_2$ for all $x \in M$, the map $J: x \to x^*$ can be uniquely extended to a conjugate linear isometry on $L^2(M)$, still denoted by J. From the theory of W* algebras it is well-known that $\lambda(M)' = J\lambda(M)J$ and $\lambda(M) = J\lambda(M)'J$. Similarly, to Theorem 1 and Corollary 1 we have the following assertion.

Theorem 2

The map $\beta': \lambda(M)' \to \lambda(M)'$ defined as $\beta'(\cdot) = J\beta(J \cdot J)J$, is an involutive *-antiautomorphism of $\lambda(M)'$. The set $\lambda_r(M, \alpha)' = \{\lambda_{x'} \in \lambda(M)': \beta'(\lambda_{x'}) = \lambda_{x'}^*\}$ is a real W^*-algebra, and $\lambda(M)'$ is the complexification of $\lambda_r(M, \alpha)'$, i.e. $\lambda_r(M, \alpha)' + i\lambda_r(M, \alpha)' = \lambda(M)'$.

We have the following connection between $\lambda_r(M, \alpha)$ and $\lambda_r(M, \alpha)'$.

Theorem 3

$\lambda_r(M,\alpha)' = J\lambda_r(M,\alpha)J.$

Proof

Since $\lambda_x \in \lambda_r(M,\alpha)$ implies that $J\lambda_x J \in J\lambda_r(M,\alpha)J$ and $\beta(\lambda_x) = \lambda_x^*$, we have

$$\beta'(J\lambda_x J) = J\beta(JJ\lambda_x JJ)J = J\beta(\lambda_x)J = J\lambda_x^* J = (J\lambda_x J)^*.$$

Hence $J\lambda_x J \in \lambda_r(M,\alpha)'$, i.e. $J\lambda_r(M,\alpha)J \subset \lambda_r(M,\alpha)'$. Conversely, let $\lambda_{x'} \in \lambda_r(M,\alpha)' \subset \lambda(M)' = J\lambda(M)J$. Then $\lambda_{x'} = J\lambda_y J$, for some $\lambda_y \in \lambda(M)$. Since $\beta(\lambda_{x'}) = \lambda_{x'}^*$, we have $\beta'(J\lambda_y J) = J\lambda_y^* J$, i.e. $J\beta(JJ\lambda_y JJ)J = J\lambda_y^* J$. Hence $J^2\beta(\lambda_y)J^2 = J^2\lambda_y^* J^2$, i.e. $\beta(\lambda_y) = \lambda_y^*$ 'Therefore $\lambda_y \in \lambda_r(M,\alpha)$. Thus we obtain $\lambda_{x'} = J\lambda_y J = J\lambda_r(M,\alpha)J$, and therefore $\lambda_r(M,\alpha)' \subset J\lambda_r(M,\alpha)J$

Theorem 4

The real W^*-algebra $\lambda_r(M,\alpha)'$ is the commutant of $\lambda_r(M,\alpha)$ in the algebra $B_r(L^2(M,\alpha))$, i.e. $\lambda_r(M,\alpha)' = \{\lambda_x \in B_r(L^2(M,\alpha)) : \lambda_x\lambda_y = \lambda_y\lambda_x, \forall\lambda_y \in \lambda_r(M,\alpha)\}$

Proof

Similarly to the proof of Theorem 1 for $\beta'(\lambda_x) = \lambda_x^*$ we have $\lambda_x \in B_r(L^2(M,\alpha))$. Therefore $\lambda_r(M,\alpha)' \subset B_r(L^2(M,\alpha))$. On the other hand for any $\lambda_x \in \lambda_r(M,\alpha)' \subset \lambda(M)'$ and $\lambda_y \in \lambda_r(M,\alpha) \subset \lambda(M)$, we have $\lambda_x\lambda_y = \lambda_y\lambda_x$.

Main Results

Theorem 5

Let $M_1 \subset B(H_1)$ and $M_2 \subset B(H_2)$ be two W^*-algebras and let α_i be an involutive *-antiautomorphism of M_i, $i = 1,2$. If $\Phi: M_1 \to M_2$ is a normal *-homomorphism with $\Phi \circ \alpha_1 = \alpha_2 \circ \Phi$, then

$$\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1,$$

where

1) Φ_1 is a * homomorphism from M_1 onto $M_1 \bar{\otimes} \mathbb{C}\mathbf{1}_L$ with $\Phi_1 \circ \alpha_1 = \tilde{\alpha}_1 \circ \Phi_1$ defined as $\Phi(a) = a \otimes \mathbf{1}_L$, where $\mathbf{1}_L$ is the identity operator on an appropriate Hilbert space L and $\tilde{\alpha_1} = \alpha_1 \otimes \mathrm{id}$;

2) Φ_2 is a *-homomorphism from $M_1 \bar{\otimes} \mathbb{C}\mathbf{I}_L$ onto $(M_1 \bar{\otimes} \mathbb{C}\mathbf{1}_L)p'$ with $\Phi_2 \circ \tilde{\alpha}_1 = \bar{\alpha}_1 \circ \Phi_2$ defined as $\Phi_2(a \otimes \mathbf{1}_L) = (a \otimes \mathbf{1}_L)p'$, where p' is a projection from $(M_1 \bar{\otimes} \mathbb{C}\mathbf{1}_L)'$ with $\tilde{\alpha_1}'(p') = p'$ and $\tilde{\alpha_1}' = J_1\tilde{\alpha_1}(J_1(.)J_1)J_1 \otimes \mathrm{id}$, $\bar{\alpha}_1(\cdot\, p') = \tilde{\alpha_1}(\cdot)p'$;

3) Φ_3 is a *-isomorphism from $(M_1 \bar{\otimes} \mathbb{C}\mathbf{1}_L)p'$ to M_2 with $\Phi_3 \circ \tilde{\alpha}_1 = \alpha_2 \circ \Phi_3$.

Proof

First we assume that (M) H_2^r is a real Hilbert space and

$$\overline{M_2\eta} = \overline{(M_2,\alpha_2)\eta} + i\overline{(M_2,\alpha_2)\eta} = H_2^r + iH_2^r = H_2,$$

hence η is a cyclic vector of M_2. Since $\Phi \circ \alpha_1 = \alpha_2 \circ \Phi$, for all $a \in (M_1,\alpha_1)$ we have $\alpha_2(\Phi(a)) = \Phi(\alpha_1(a)) = \Phi(a^*) = \Phi(a)^*$, i.e. $\Phi(a) \in (M_2,\alpha_2)$. Hence $\Phi\big((M_1,\alpha_1)\big) \subset (M_2,\alpha_2)$. Define a functional φ by

$$\varphi(a) = \langle \Phi(a)\eta, \eta\rangle,\ a \in (M_1,\alpha_1).$$

Obviously, φ is a normal positive functional on (M_1,α_1). We can extend φ by linearity to a functional on M_1 (still denoted by φ) such that

$$\varphi(a+ib) = \varphi(a) + i\varphi(b),\ a,b \in (M_1,\alpha_1),$$

which clearly also is a normal positive functional. Let H_1^r be a real Hilbert space with $H_1^r + iH_1^r = H_1$ such that $(M_1,\alpha_1) \subset B(H_1^r)$. By [8,4.2.1] there is a sequence $(\xi_n) \subset H_1^r$ with $\sum_n \|\xi_n\|^2 < \infty$ such that $\varphi(a) = \sum_n < a\xi_n, \xi_n >$, for all $a \in (M_1,\alpha_1)$. Set $L_r = \ell_2^r = \{(x_n) \subset \mathbb{R}: \sum_n x_n^2 < \infty\}, L = L_r + iL_r, \xi = (\xi_n) \subset H_1^r \otimes L_r$ and $\Phi_1(a) = a \otimes \mathbf{1}_L$ for all $a \in M_1$. Then Φ_1 is a map from M_1 to $M_1 \bar{\otimes} \mathbb{C}\mathbf{1}_L$ and

$$\begin{aligned}(\Phi_1 \circ \alpha_1)(a) &= \Phi_1(\alpha_1(a)) = \alpha_1(a) \otimes \mathbf{1}_L = (\alpha_1 \otimes id)(a \otimes \mathbf{1}_L)\\ &= \tilde{\alpha_1}(\Phi_1(a)) = (\tilde{\alpha_1} \circ \Phi_1)(a)\end{aligned}$$

i.e. $\Phi_1 \circ \alpha_1 = \tilde{\alpha_1} \circ \Phi_1$. Moreover, for all $a \in (M_1,\alpha_1)$ we have

$$\langle \Phi_1(a)\xi, \xi\rangle = \langle (a \otimes \mathbf{1}_{L_r})\xi, \xi\rangle = \sum_n \langle a\xi_n, \xi_n\rangle = \varphi(a)$$

Let p' be the projection from $H_1^r \otimes L_r$ to $\overline{\Phi_1\big((M_1,\alpha_1)\big)\xi}$. Then for all $x = a \otimes \mathbf{1}_{L_r} \in \big((M_1,\alpha_1) \bar{\otimes} \mathbb{R}\mathbf{1}_{L_r}\big)$ we have

$$\begin{aligned}(p'x)\xi &= p'\left((a \otimes \mathbf{1}_{L_r})\xi\right) = p'(\Phi_1(a)\xi) = \Phi_1(a)\xi\\ &= (a \otimes \mathbf{1}_{L_r})\xi = x\xi = x\left((\mathbf{1} \otimes \mathbf{1}_{L_r})\xi\right) = x(\Phi_1(\mathbf{1})\xi)\\ &= x\big(p'(\Phi_1(\mathbf{1})\xi)\big) = x(p'(\xi)) = (xp')\xi\end{aligned}$$

Similarly, for all $\gamma \in H_1^r \otimes L_r$ with $\gamma \neq \xi$ we also obtain

$$\begin{aligned}(p'x)\gamma &= p'(\Phi_1(a)\gamma) = \theta = x(\theta) = x\big(p'(\Phi_1(\mathbf{1})\gamma)\big)\\ &= xp'\left((\mathbf{1} \otimes \mathbf{1}_{L_r})\gamma\right) = xp'(\gamma)\end{aligned}$$

Therefore $p'x = xp'$, i.e. $p' \in \left((M_1,\alpha_1) \bar{\otimes} \mathbb{R}\mathbf{1}_{L_r}\right)'$. Hence $p' \in \left(M_1 \bar{\otimes} \mathbb{C}\mathbf{1}_L\right)'$ and for $\tilde{\alpha_1}' = J_1\tilde{\alpha_1}(J_1(.)J_1)J_1 \otimes id$ we have $\overline{\alpha_1}'(p') = p'$.

Define the map $\Phi_2: M_1 \bar{\otimes} \mathbb{C}\mathbf{1}_L \to \left(M_1 \bar{\otimes} \mathbb{C}\mathbf{1}_L\right)p'$ as $\Phi_2(a \otimes \mathbf{1}_L) = (a \otimes \mathbf{1}_L)p', a \in M_1$. Then

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$$\begin{aligned}(\Phi_2 \circ \tilde{\alpha}_1)(a \otimes \mathbf{1}_L) &= \Phi_2\big(\tilde{\alpha}_1(a \otimes \mathbf{1}_L)\big) = \Phi_2(\alpha_1(a) \otimes \mathbf{1}_L) \\ &= (\alpha_1(a) \otimes \mathbf{1}_L)p' = \tilde{\alpha}_1(a \otimes \mathbf{1}_L)p' \\ &= \bar{\alpha}_1\big((a \otimes \mathbf{1}_L)p'\big) = \bar{\alpha}_1\big(\Phi_2(a \otimes \mathbf{1}_L)\big) \\ &= (\bar{\alpha}_1 \circ \Phi_2)(a \otimes \mathbf{1}_L)\end{aligned}$$

hence $\Phi_2 \circ \tilde{\alpha}_1 = \bar{\alpha}_1 \circ \Phi_2$. Since $p'\xi = p'\big((\mathbf{1} \otimes \mathbf{1}_L)\xi\big) = p'(\Phi_1(\mathbf{1})\xi) = \Phi_1(\mathbf{1})\xi = \xi$, we have

$$\begin{aligned}\langle(\Phi_2 \circ \Phi_1)(a)\xi, \xi\rangle = &\ \Big\langle\Big(\Phi_2\big(a \otimes \mathbf{1}_{L_r}\big)\Big)\xi, \xi\Big\rangle = \big\langle\big(a \otimes \mathbf{1}_{L_r}\big)p'\xi, \xi\big\rangle \\ = &\ \big\langle\big(a \otimes \mathbf{1}_{L_r}\big)\xi, \xi\big\rangle = \langle\Phi_1(a)\xi, \xi\rangle = \varphi(a),\end{aligned}$$

for all $a \in (M_1, \alpha_1)$, i.e. $\varphi(a) = \langle(\Phi_2 \circ \Phi_1)(a)\xi, \xi\rangle$. Now, define a linear map $u: \Phi\big((M_1, \alpha_1)\big)\eta \to p'(H_1^r \otimes L_r)$ as follows:

$$u\Phi(a)\eta = (\Phi_2 \circ \Phi_1)(a)\xi = p'(a\xi_n) = (a\xi_n)\ \big(a \in (M_1, \alpha_1)\big).$$

Since $u\Phi(a)\eta = (\Phi_2 \circ \Phi_1)(a)\xi$ and $\langle\Phi(a)\eta, \eta\rangle = \varphi(a) = \langle(\Phi_2 \circ \Phi_1)(a)\xi, \xi\rangle$ $\big(a \in (M_1, \alpha_1)\big)$, it follows that $\| u\Phi(a)\eta \|' = \| \Phi(a)\eta \|_2^r$, i.e. the map u is an isometry, where $\|\cdot\|_2^r$ is the norm of the space H_2 and $\|\cdot\|'$ is the norm of the space $H_1^r \otimes L_r$. Moreover, since $\Phi\big((M_1, \alpha_1)\big)\eta = (M_2, \alpha_2)\eta$, $(\Phi_2 \circ \Phi_1)\big((M_1, \alpha_1)\big)\xi = \Phi_1\big((M_1, \alpha_1)\big)\xi$ and

$$\overline{\Phi\big((M_1, \alpha_1)\big)\eta} = \overline{(M_2, \alpha_2)\eta} = H_2^r$$

$$\big((M_1, \alpha_1)\big)\xi = \overline{\Phi_1\big((M_1, \alpha_1)\big)\xi} = p'(H_1^r \otimes L_r)$$

u can be extended to a unitary operator $\bar{u}: H_2^r \to p'(H_1^r \otimes L)$. Clearly,

$$\bar{u}\Phi(a)\bar{u}^{-1} = \Phi_2 \circ \Phi_1(a),\ a \in (M_1, \alpha_1).$$

Therefore we can define a spatial real $*$-isomorphism $\Phi_3: \Big((M_1, \alpha_1) \bar{\otimes} \mathbb{R}\mathbf{1}_{L_r}\Big)p' \to (M_2, \alpha_2)$ as $\Phi_3(.) = \bar{u}^{-1}(.)\bar{u}$, and it can be extended to a spatial *-isomorphism (still denoted by Φ_3)$\Phi_3: (M_1\bar{\mathbb{C}}_L)p' \to M_2$ as $\Phi_3(a + ib) = \Phi_3(a) + i\Phi_3(b)$, where $a, b \in \Big((M_1, \alpha_1) \bar{\otimes} \mathbb{R}\mathbf{1}_{L_r}\Big)p'$. Then, we have $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$.

Considering now the general case, the real Hilbert space H_2^r with $H_2^r + iH_2^r = H_2$ can be decomposed as $H_2^r = \bigoplus_l H_2^l$ and $H_2^l = \overline{(M_2, \alpha_2)\eta_l}$, where $\eta_l \in H_2^r$, for $l \in \mathbb{N}$. Let $q_l': H_2^r \to \overline{(M_2, \alpha_2)\eta_l} = H_2^l$ be the projection. Then $q_l' \in (M_2, \alpha_2)'$, for all l. For each $l, \Phi_l = q_l'\Phi: (M_1, \alpha_1) \to (M_2, \alpha_2)q_l'$ is a normal *-homomorphism, which can be extended to a normal *-homomorphism $\Phi_l: M_1 \to M_2q_l'$. Then, by the above argument $\Phi_l = \Phi_3^{(l)} \circ \Phi_2^{(l)} \circ \Phi_1^{(l)}$, for all l. Set $\Phi_i = \bigoplus_l \Phi_i^{(l)}, i = 1,2,3$. Then $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$ and the maps Φ_3, Φ_2, Φ_1 satisfy all our conditions.

Theorem 6

Let M be a finite factor and let α be an involutive * antiautomorphism of M. If $\{\pi, H\}$ is a faithful nondegenerate W^*-representation of M and $\pi \circ \alpha = \tilde{\alpha} \circ \pi$ for an involutive "-antiautomorphism $\tilde{\alpha}$ of $\pi(M)$, then there exist a projection $p' \in \left(\lambda_r(M, \alpha) \otimes \mathbf{1}_{K_r}\right)'$, and a unitary operator $u: H_r \to p'(L^2(M, \alpha) \otimes K_r)$ such that

$$u\pi(x) = (\lambda(x) \otimes \mathbf{1}_K)u, \ x \in M,$$

i.e., the real W^*-algebras $\pi(M, \alpha)(= (\pi(M), \tilde{\alpha}))$ and $\left(\lambda_r(M, \alpha) \otimes \mathbf{1}_{K_r}\right)p'$ are spatially *-isomorphic and therefore the W^*-algebras $\pi(M)$ and $(\lambda(M) \otimes \mathbf{1}_K)p'$ are also spatially * isomorphic; where K_r is a separable infinite dimensional Hilbert space, and $K = K_r + iK_r$

Proof

Set $M_1 = \lambda(M)$ and $M_2 = \pi(M)$. Define the map $\Phi: M_1 \to M_2$ by $\Phi(\lambda(x)) = \pi(x)$. Then Φ is a *-isomorphism and $\Phi(\lambda_r(M, \alpha)) \subset (\pi(M), \bar{\alpha})$. Now the conclusion follows immediately from Theorem 5 and the separability of H.

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